

# INJECTIVE MODULES OVER DOWN-UP ALGEBRAS

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**ABSTRACT.** The purpose of this paper is to study finiteness conditions on injective hulls of simple modules over Noetherian Down-Up algebras. We will show that the Noetherian Down-Up algebras  $A(\alpha, \beta, \gamma)$  which are fully bounded are precisely those which are module-finite over a central subalgebra. We show that injective hulls of simple  $A(\alpha, \beta, \gamma)$ -modules are locally Artinian provided the roots of  $X^2 - \alpha X - \beta$  are distinct roots of unity or both equal to one.

*dedicated to Patrick F. Smith - teacher and friend*

## 1. INJECTIVE HULLS OF SIMPLE MODULES OVER NOETHERIAN RINGS

Injective modules are the building blocks in the theory of Noetherian rings. Matlis showed that any indecomposable injective module over a commutative Noetherian ring is isomorphic to the injective hull  $E(R/P)$  of some prime factor ring of  $R$ . He also showed that any injective hull of a simple module is Artinian (see [16] and [17, Proposition 3]). Vamos characterized commutative rings  $R$  whose injective hulls of simples are Artinian as those whose localization  $R_M$  by maximal ideals are Noetherian ([21, Theorem 2]). Not necessarily commutative rings whose injective hulls of simples are Artinian were studied by Jans ([7]) and termed *co-Noetherian*. In [6] Dahlberg showed that injective hulls of simple modules over  $U(\mathfrak{sl}_2)$  are locally Artinian. Since  $U(\mathfrak{sl}_2)$  is an instance of a larger class of Noetherian domains, the Down-up Algebras, introduced by Benkart and Roby in [2], Patrick F. Smith asked which Noetherian Down-Up algebras satisfy this finiteness condition on their injective hulls of simple modules. The purpose of this note is to give a partial answer to Smith's question. Recall that a module is called *locally Artinian* if every of its finitely generated submodules is Artinian.

1.1. In connection with the Jacobson Conjecture for Noetherian rings Jategaonkar showed in [9] (see also [5, 20]) that the injective hulls of simple modules are locally Artinian provided the ring  $R$  is fully bounded Noetherian.

1.2. Consider the following property for a ring  $A$ :

( $\diamond$ ) Injective hulls of simple right  $A$ -modules are locally Artinian.

Property ( $\diamond$ ) is obviously equivalent to the condition, that all finitely generated essential extensions of simple right  $A$ -modules are Artinian. And in case  $A$  is right Noetherian property ( $\diamond$ ) is further equivalent to the class of semi-Artinian right  $A$ -modules, i.e. modules  $M$  that are the union of their socle series, to be closed under essential extensions. In torsion theoretic terms,  $A$  has property ( $\diamond$ ) if and only if the hereditary torsion theory generated by all simple right  $A$ -modules is stable.

1.3. Let us first explain, why every commutative Noetherian ring has property ( $\diamond$ ) without using Matlis result. The Artin-Rees Lemma says (in one of its versions) that any ideal  $I$  of a commutative Noetherian ring  $A$  has the Artin-Rees property, i.e. for any essential extension  $N \subseteq M$  of finitely generated  $A$ -modules with  $NI = 0$ , there exists  $n > 0$  such that  $MI^n = 0$ . Thus  $M$  has a finite filtration

$$0 \subseteq MI^{n-1} \subseteq MI^{n-2} \subseteq \dots \subseteq MI^2 \subseteq MI \subseteq M$$

such that each of its factors  $MI^{k-1}/MI^k$  is a finitely generated  $A/I$ -module. If  $N$  is a simple right  $A$ -module and  $I = \text{Ann}_A(N)$  then  $A/I$  is a field, hence Artinian, and so any factor in the filtration of  $M$  is Artinian, making  $M$  an Artinian module.

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A sufficient condition for  $(\diamond)$  is therefore that right primitive ideals of  $A$  have the Artin-Rees property and that primitive factor rings of  $A$  are Artinian.

1.4. For simple Noetherian algebras, the above argument cannot be used due to the absence of non-zero proper ideals. However if  $A$  is a (not necessarily commutative) semiprime Noetherian ring of Krull dimension one, then for any essential right ideal  $I$  of  $A$ , the Krull dimension of  $A/I$  is one lower than the Krull dimension of  $A$ , hence Artinian. For any extensions  $E \subseteq M$  of a simple right  $A$ -module  $E$  by a cyclic right  $A$ -module  $M = A/I$ , we have  $E \simeq J/I$  with  $J/I$  essential in  $A/I$ . Since pre-images of essential submodules are essential, also  $J$  is essential in  $A$ . Thus  $M/E \simeq A/J$  is Artinian and  $M$  being an extension of the two Artinian modules  $E$  and  $M/E$  is also Artinian. Thus any semiprime Noetherian ring of Krull dimension one has property  $(\diamond)$ . This applies in particular to the first Weyl algebra  $A_1(\mathbb{C}) = \mathbb{C}[x][y; \partial/\partial x]$ .

1.5. Let  $E \subseteq M$  be an essential extension of a simple right  $A$ -module  $E$  by a Noetherian module  $M$ . Let  $P = \text{Ann}_A(E)$  be its annihilator. Suppose there exists a non-zero central element  $x \in P \cap Z(A)$ , i.e.  $Ex = 0$ . Denote by  $f : M \rightarrow M$  the  $A$ -linear map  $f(m) = mx$ . Its kernel is  $\text{Ker}(f) = \text{Ann}_M(x)$ . By Fitting's Lemma there exists a number  $n > 0$  such that  $\text{Im}(f^n) \cap \text{Ker}(f)^n = 0$ . Since  $M$  is uniform and  $E \subseteq \text{Ker}(f)^n$  is non-zero, we have  $\text{Im}(f^n) = Mx^n = 0$ . Hence we have again a finite filtration

$$0 \subseteq \text{Ker}(f) = \text{Ann}_M(x) \subseteq \text{Ker}(f)^2 \subseteq \cdots \subseteq \text{Ker}(f)^{n-1} \subseteq \text{Ker}(f)^n = M$$

whose factors  $\text{Ker}(f)^k / \text{Ker}(f)^{k-1}$  are  $A/Ax$ -modules and embed into  $\text{Ann}_M(x)$ , because  $f$  induces monomorphisms

$$M / \text{Ker}(f)^{n-1} \hookrightarrow \text{Ker}(f)^{n-1} / \text{Ker}(f)^{n-2} \hookrightarrow \cdots \hookrightarrow \text{Ker}(f)^2 / \text{Ker}(f) \hookrightarrow \text{Ker}(f).$$

Hence  $M$  is Artinian if and only if  $\text{Ann}_M(x) = \text{Ker}(f)$  is Artinian. The same argument also works for  $x$  being a normal element. In this case  $f$  is not  $A$ -linear anymore, but preserves submodules (see [8, Lemma 2]).

1.6. The last subsection allow us now to state the following reduction of our problem, in case  $A$  has a non-trivial center.

**Proposition.** *The following statements are equivalent for a countably generated Noetherian algebra  $A$  with Noetherian center over an algebraically closed uncountable field  $K$ .*

- (a) *Injective hulls of simple right  $A$ -modules are locally Artinian;*
- (b) *Injective hulls of simple right  $A/\mathfrak{m}A$ -modules are locally Artinian for all maximal ideals  $\mathfrak{m}$  of  $Z(A)$ .*

*Proof.* (a)  $\Rightarrow$  (b) is clear, since property  $(\diamond)$  is inherited by factor rings.

(b)  $\Rightarrow$  (a): First note that the Proposition is void if  $A$  has trivial center  $Z(A) = K$ . Hence we will suppose  $Z(A) \neq K$ . Moreover note, that any countably generated algebra  $A$  over an uncountable field  $K$  has the endomorphism property ([10, 9.1.8]). Hence the endomorphism ring of each simple right  $A$ -module  $E$  is  $\text{End}_A(E) \simeq K$  as  $K$  was supposed to be algebraically closed. Let  $E$  be a simple right  $A$ -module and  $M$  be a finitely generated essential extension of  $E$ . Denote  $P = \text{Ann}_A(E)$  and  $\mathfrak{m} = P \cap Z(A)$  which is a maximal ideal of  $Z(A)$  as the  $A$ -action on  $E$  restricts to an  $Z(A)$ -action on  $E$  which is not faithful as  $\text{End}_A(E) = K$  and  $Z(A) \neq K$ . As  $A$  and hence  $Z(A)$  are Noetherian, there exist central elements  $x_1, \dots, x_k$  that generate  $\mathfrak{m}$ . By 1.5  $M$  is Artinian if and only if  $M_1 = \text{Ann}_M(x_1)$  is Artinian. Applying the same argument again leads to  $M$  Artinian if and only if  $M_2 = \text{Ann}_{M_1}(x_2) = \text{Ann}_M(\{x_1, x_2\})$  Artinian. Iterating  $k$  times leads to  $M$  being Artinian if and only if  $\text{Ann}_M(\{x_1, \dots, x_k\}) = \text{Ann}_M(\mathfrak{m})$  being Artinian. Since  $E \subseteq \text{Ann}_M(\mathfrak{m})$  is an essential extension of  $A/\mathfrak{m}A$ -modules, with  $\text{Ann}_M(\mathfrak{m})$  being finitely generated, we get by hypothesis (b), that  $\text{Ann}_M(\mathfrak{m})$  is Artinian.  $\square$

1.7. Let  $\mathfrak{h}$  be the 3-dimensional Heisenberg algebra over  $\mathbb{C}$  which is generated by  $x, y, z$  with Lie algebra structure given by  $[x, y] = z$  and  $[x, z] = 0 = [y, z]$ . Let  $A = U(\mathfrak{h})$ . Then  $Z(A) = \mathbb{C}[z]$  and its maximal ideals are of the form  $\mathfrak{m}_\lambda = \langle z - \lambda \rangle$ , with  $\lambda \in \mathbb{C}$ . For  $\lambda = 0$ , we have that  $A/\mathfrak{m}_0 A \simeq \mathbb{C}[x, y]$  is a commutative Noetherian domain and hence has property  $(\diamond)$ . For  $\lambda \neq 0$ , we have  $A/\mathfrak{m}_\lambda A \simeq \mathbb{C}[x][y; \partial/\partial x]$  is the first Weyl algebra, which is a Noetherian domain of Krull dimension 1 (see [10, 6.6.15]) and hence also has property  $(\diamond)$  by 1.4. Hence by Proposition 1.6 we have that  $U(\mathfrak{h})$  has the property  $(\diamond)$ .

1.8. In contrast to the Heisenberg algebra, which is a nilpotent Lie algebra, Ian Musson showed that no non-nilpotent soluble finite dimensional complex Lie algebra  $\mathfrak{g}$  has property  $(\diamond)$ , i.e. there exists a non-Artinian finitely generated essential extension of a simple right  $U(\mathfrak{g})$ -module ([18]).

1.9. In [6] Dahlberg showed that  $U(\mathfrak{sl}_2)$  has property  $(\diamond)$ . Since  $U(\mathfrak{sl}_2)$  and  $U(\mathfrak{h})$  are two instances of a larger class of Noetherian domains, the Down-up Algebras, introduced by Benkart and Roby ([2]) we ask which Noetherian Down-Up algebras satisfy property  $(\diamond)$ .

In the following section we will recall the definition of Down-up algebras and determine when they are fully bounded Noetherian. In the last section we show that some of the Noetherian Down-Up algebras of Krull dimension 2 have property  $(\diamond)$ . For simplicity all algebras are considered to be algebras over the complex numbers  $\mathbb{C}$ .

## 2. FULLY BOUNDED NOETHERIAN DOWN-UP ALGEBRAS

The down-up algebras form a three parameter family of associative algebras. For any parameter set  $(\alpha, \beta, \gamma) \in \mathbb{C}^3$  one defines a  $\mathbb{C}$ -algebra, denoted by  $A = A(\alpha, \beta, \gamma)$ , generated by two elements  $u$  and  $d$  subject to the relations

$$\begin{aligned} d^2 u &= \alpha d u d + \beta u d^2 + \gamma d \\ d u^2 &= \alpha u d u + \beta u^2 d + \gamma u \end{aligned}$$

2.1. Kirkman, Musson and Passman proved that  $A$  is noetherian if and only if it is a domain if and only if  $\beta \neq 0$  if and only if  $\mathbb{C}[ud, du]$  is a polynomial ring (see [12]).

2.2. Any Noetherian down-up algebra can be represented as generalized Weyl algebra. Let  $R$  be a commutative ring and  $\sigma$  an automorphism of  $R$  and  $x$  an element of  $R$ , the generalized Weyl algebra is the algebra  $R(\sigma, x)$  generated over  $R$  in two indeterminates  $u, d$  subject to the relations:  $ur = \sigma(r)u, dr = \sigma^{-1}(r)d$  for  $r \in R$  and  $ud = x, du = \sigma^{-1}(x)$ . In other words

$$R(\sigma, x) := R\langle u, d \rangle / \langle ur - \sigma(r)u, dr - \sigma^{-1}(r)d, ud = x, du = \sigma^{-1}(x) \forall r \in R \rangle$$

As shown in [12] and [14], if  $\beta \neq 0$ , then  $A \simeq R(\sigma, x)$  where  $R = \mathbb{C}[x, y]$ ,  $\sigma(x) = \frac{y - \alpha x - \gamma}{\beta}$  and  $\sigma(y) = x$ . The isomorphism maps  $ud$  to  $x$  and  $du$  to  $y$ . Kulkarni calls a Noetherian down-up algebra a down-up algebra at roots of unity if the associated automorphism  $\sigma$  has finite order.

2.3. The center of  $R(\sigma, x)$  is generated by the fix ring  $R^\sigma$  and the elements  $u^m, d^m$  where  $m$  is the order of  $\sigma$  or 0 if the order is infinite (see [14, 2.0.1]). Hence if  $\sigma$  has finite order, then  $R(\sigma, x)$  is generated by  $1, u, \dots, u^{m-1}, d, \dots, d^{m-1}$  as a module over its center. On the other hand if  $\sigma$  has infinite order, then the center of  $R(\sigma, x)$  is equal to the fix ring  $R^\sigma$ . Hence  $R(\sigma, x)$  cannot be finitely generated over a central subalgebra since otherwise it would be also finitely generated as a module over  $R$  which is impossible since  $R(\sigma, x) = \bigoplus_{n \in \mathbb{Z}} A_n$  is  $\mathbb{Z}$ -graded with  $A_n = Ru^n$  and  $A_{-n} = Rd^n$  for  $n > 0$  and  $A_0 = R$ . As any finitely generated  $R$ -submodule of  $R(\sigma, x)$  is bounded and  $A_n \neq 0$  for all  $n$ ,  $R(\sigma, x)$  is not finitely generated over  $R$ . Thus we proved:

**Lemma.**  $R(\sigma, x)$  is module-finite over a central subalgebra if and only if  $\sigma$  has finite order.

2.4. Recall that a ring  $R$  is called right (resp. left) bounded if every right (resp. left) essential ideal contains a non-zero two-sided ideal.  $R$  is called right fully bounded noetherian if it is right Noetherian and every prime factor ring is right bounded. As mentioned in the first section, fully bounded Noetherian rings have property  $(\diamond)$ . The considerations above deduce now the following characterization of Down-up algebras at roots of unity.

**Theorem.** *The following statements are equivalent for a Noetherian Down-up algebra  $A = A(\alpha, \beta, \gamma)$ :*

- (1)  *$A$  is a down-up algebra at roots of unity;*
- (2)  *$A$  is module-finite over a central subalgebra;*
- (3)  *$A$  satisfies a polynomial identity;*
- (4)  *$A$  is fully bounded Noetherian;*
- (5) *The roots of the polynomial  $X^2 - \alpha X - \beta$  are distinct roots of unity such that both are also different from 1 if  $\gamma \neq 0$ .*

*Proof.* (1)  $\Leftrightarrow$  (2) follows from Lemma 2.3. (2)  $\Rightarrow$  (3) any module-finite algebra over a commutative subalgebra satisfies a polynomial identity (see for instance [10, 13.4.9]).

(3)  $\Rightarrow$  (4) any Noetherian algebra that satisfies a polynomial identity is fully bounded Noetherian (see for instance [10, 13.6.6]).

(4)  $\Rightarrow$  (1) we will show that  $A$  is a down-up algebra at roots of unity.

Note that by [4, p. 287] any Noetherian Down-up algebra  $A$  can be embedded into the skew Laurent polynomial ring  $R[z, z^{-1}; \theta]$  where  $R = \mathbb{C}[x, y]$ ,  $\theta(x) = y$  and  $\theta(y) = \alpha y + \beta x + \gamma$ . As a right  $R$ -module  $R[z, z^{-1}; \theta]$  is free on the basis  $\{z^n \mid n \in \mathbb{N}\}$  and the multiplication in  $S$  is defined by  $rz = z\theta(r)$  for  $r \in R$ . The embedding  $\iota : A \rightarrow R[z, z^{-1}; \theta]$  is given by  $\iota(d) = z^{-1}$  and  $\iota(u) = xz$ , so that  $\iota(ud) = x$  and  $\iota(du) = y$ . By the proof of [4, Lemma 1.2],  $R[z, z^{-1}; \theta]$  is the localization of  $A$  by the Ore set  $\{d^n \mid n \in \mathbb{N}\}$ . By [3, 4.1.8] (or by [5, Proposition 1.5] and [13, Theorem 3.5]), if  $A$  is fully bounded Noetherian, then also  $A_d$ , hence  $R[z, z^{-1}; \theta]$ . By [3, Proposition 4.1.12],  $\theta$  must have finite order. Since  $A \simeq A' = A(-\alpha\beta^{-1}, \beta^{-1}, -\gamma\beta^{-1})$  by [4, Lemma 4.1], also  $\theta'$  has finite order, where  $\theta'$  is the automorphism of  $\mathbb{C}[x, y]$  defined analogously by  $\theta'(x) = y$  and  $\theta'(y) = (-\alpha\beta^{-1})y + (-\beta^{-1})x + (-\gamma\beta^{-1})$ . Denoting by  $\tau$  the automorphism of  $\mathbb{C}[x, y]$  which interchanges  $x$  and  $y$ , we have that  $\sigma = \tau\theta'\tau$  has finite order, where  $\sigma$  equals the automorphism that represents  $A$  as a generalized Weyl algebra as in 2.2. Hence  $A$  is a down-up algebra at roots of unity.

(4)  $\Leftrightarrow$  (5) Let  $r_1, r_2$  be the roots of the polynomial  $X^2 - \alpha X - \beta$  and let  $\theta$  be the automorphism that defines the skew Laurent ring  $R[z, z^{-1}; \theta]$  as above, with  $R = \mathbb{C}[x, y]$ . Note that  $\theta$  stabilizes the vector space  $V$  spanned by  $1, x$  and  $y$ . In [4, p.288-289] a basis  $1, w_1, w_2$  for  $V$  had been found such that the matrix of  $\theta$  with respect to this basis is in Jordan canonical form. Four cases had to be considered: if both roots  $r_1$  and  $r_2$  are different and also different from 1, then there exists such a basis such that  $\theta(w_i) = r_i w_i$  for  $i = 1, 2$ . Hence  $\theta$  has finite order if and only if both roots are roots of unity.

If  $r_1 = 1$  and  $r_2 \neq 1$ , then there exists a basis such that  $\theta(w_1) = w_1 + \gamma$  and  $\theta(w_2) = r_2 w_2$ . Hence  $\theta$  has finite order if and only if  $\gamma = 0$  and  $r_2$  is a root of unity.

If both roots are the same  $r = r_1 = r_2$  but different from 1, then there exists a basis such that  $\theta(w_1) = r w_1$  and  $\theta(w_2) = r w_2 + w_1$ . Hence for any  $n$ ,  $\theta^n(w_1) = r^n w_1$  and  $\theta^n(w_2) = r^n w_2 + n r^{n-1} w_1$ . Hence  $\theta$  cannot have finite order. Similarly, if both roots are 1, there exists a basis such that  $\theta(w_1) = w_1 + \gamma$  and  $\theta(w_2) = w_2 + w_1$  that implies that  $\theta$  will not have finite order.  $\square$

### 3. NON-PRIMITIVE DOWN-UP ALGEBRAS OF KRULL DIMENSION TWO

A theorem of Bavula and Lenagan states, that the Krull dimension of  $A = A(\alpha, \beta, \gamma)$  is 2 if and only if  $\alpha + \beta = 1$  and  $\gamma \neq 0 \neq \beta$ ; otherwise the Krull dimension is 3 (see [1, Theorem 4.2]). Equivalently  $A$  has Krull dimension 2 precisely if  $\gamma, \beta \neq 0$  and 1 is a root of  $X^2 - \alpha X - \beta$ . We will focus in this section on those Down-Up algebras which are denoted by  $A_\eta$  in [4]:  $A_\eta := A(1 + \eta, -\eta, 1)$  for  $\eta \in \mathbb{C}^\times =: \mathbb{C} \setminus \{0\}$ .

3.1. By [22, Theorem 1.3(g)] the center of the algebras  $A_\eta$ , with  $\eta$  being a primitive  $N$ -th root of unity different from 1, is a polynomial ring  $\mathbb{C}[\omega]$  in one variable, where  $\omega$  is the element  $\omega = z^N$  with  $z = du - ud + \frac{\gamma}{\eta-1}$ . Note that if  $\eta$  is not a root of unity, then the center of  $A_\eta$  is trivial. We will apply Proposition 1.6 to prove the following:

**Theorem.**  $A_\eta$  satisfies  $(\diamond)$  if  $\eta$  is a root of unity different from 1.

*Proof.* Let  $\eta$  be a primitive  $N$ th root of unity different from 1. We intend to use Proposition 1.6. As mentioned before  $Z(A_\eta) = \mathbb{C}[\omega]$ . The maximal ideals of  $\mathbb{C}[\omega]$  are of the form  $\langle \omega - c \rangle$  with  $c \in \mathbb{C}$ . By [19, Theorem 8.1(C1)] any ideal of the form  $(\omega - c)A_\eta$  with  $c \in \mathbb{C}^\times$  is right primitive. Hence  $A_\eta/(\omega - c)A_\eta$  is a right primitive Noetherian ring of Krull dimension 1 and hence has property  $(\diamond)$  by 1.4. For  $c = 0$ , let  $B = A_\eta/\omega A_\eta$ . We have that  $\omega = z^N$  with  $z = du - ud + \frac{\gamma}{\eta-1}$ . As  $z$  is a normal element of  $A_\eta$  it is also normal in  $B$ . By [19, Theorem 8.1(C1)]  $zA_\eta$  is a primitive right ideal of  $A_\eta$  and hence also of  $B$ . Thus  $B/zB$  is a primitive Noetherian ring of Krull dimension 1 and has property  $(\diamond)$  again by 1.4. Given any essential extension  $E \subseteq M$  of finitely generated  $B$ -modules, with  $E$  being simple, we first note, that  $zE = 0$ , since otherwise  $E = zE = \cdots = z^N E = \omega E = 0$  - a contradiction. Since  $B/zB$  satisfies  $(\diamond)$ ,  $\text{Ann}_M(z)$  is Artinian and by [8, Lemma 2]  $M$  is Artinian.

This shows that any factor  $A_\eta/\mathfrak{m}A_\eta$  by a maximal ideal  $\mathfrak{m}$  of  $Z(A_\eta)$  has property  $(\diamond)$ . By Proposition 1.6  $A_\eta$  satisfies  $(\diamond)$ .  $\square$

3.2. Summarizing Theorem 2.4 and Theorem 3.1 we have the following:

**Corollary.** *The injective hull of any simple right  $A(\alpha, \beta, \gamma)$ -module is locally Artinian, if the roots of  $X^2 - \alpha X - \beta$  are distinct roots of unity or both equal to one.*

*Proof.* If the roots of  $X^2 - \alpha X - \beta$  are distinct roots of unity and also different from 1 if  $\gamma \neq 0$ , then  $A = A(\alpha, \beta, \gamma)$  is fully bounded Noetherian by Theorem 2.4. A classical result by Schelter and Jategaonkar says that the injective hull of a simple right  $R$ -module over a left Noetherian right fully bounded Noetherian ring  $R$  is locally Artinian (see for instance [11, 9.12] or [10, Proposition 6.4.14]).

Suppose  $\gamma \neq 0$  and that one of the roots is 1, then  $A \simeq A_\eta$  and Theorem 3.1 shows that  $A$  has property  $(\diamond)$ . In case both roots are 1, then  $\alpha = 2$  and  $\beta = -1$ . Since  $A_1 = A(2, -1, 1) = U(\mathfrak{sl}_2)$  and  $A(2, -1, 0) = U(\mathfrak{h})$  those algebras have property  $(\diamond)$  by [6] and 1.7.  $\square$

3.3. We were unable to find an example of a Noetherian Down-Up algebra that does not satisfy  $(\diamond)$ .

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